

Building up the elliptic flow: analytical insights

Yoshitaka Hatta¹ and Bo-Wen Xiao²

¹*Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*

²*Key Laboratory of Quark and Lepton Physics (MOE) and Institute of Particle Physics, Central China Normal University, Wuhan 430079, China*

(Dated: July 16, 2014)

In this paper, we present a fully analytical description of the early-stage formation of elliptic flow in relativistic viscous hydrodynamics. We first construct an elliptic deformation of Gubser flow which is a boost invariant solution of the Navier-Stokes equation with a nontrivial transverse profile. We then analytically calculate the momentum anisotropy of the flow as a function of time and discuss the connection with the empirical formula by Bhalerao *et al.* regarding the viscosity dependence of elliptic flow.

PACS numbers: 47.75.+f, 12.38.Mh, 11.25.Hf

I. INTRODUCTION

One of the most intriguing results of ultrarelativistic heavy ion collisions at RHIC and the LHC is the strong collectivity of the created hot and dense matter, especially the considerable elliptic flow [1–6]. In non-central collisions, the overlapping region of the colliding nuclei approximately has the shape of an ellipse in the transverse plane. This region expands hydrodynamically, and the initial anisotropy in the pressure gradient gets converted into momentum space anisotropy, resulting in the modulation of the azimuthal angle distribution of charged particles in the final state [7]

$$\frac{dN}{d\phi} \propto 1 + 2v_2 \cos 2\phi. \quad (1)$$

The coefficient v_2 is called the elliptic flow parameter and is one of the central objects of experimental and theoretical study in heavy-ion physics because it is a sensitive probe of the equilibrium/non-equilibrium properties of the created matter.

While the mechanism to generate v_2 is well understood, little is known about its analytical details. Theoretically, the extraction of v_2 relies heavily on numerical (viscous) hydrodynamic simulations supplemented with some initial condition and the equation of state (see, e.g., [8–17]). In this paper, we provide a fully analytical description of the early-stage formation of v_2 by deriving and utilizing an approximate elliptic solution of the relativistic Navier-Stokes equation. Such an analysis has long been infeasible due to the difficulty of constructing realistic elliptically-shaped solutions of hydrodynamic equations for which only a few attempts have been made in the literature [18–21]. The new solution we are able to present here has been achieved along the line of the recent progress in constructing exact solutions of viscous hydrodynamic equations using conformal symmetry [22–26]. Within the region of validity of the solution, one can study explicitly how the elliptic flow develops as a function of proper time and how the shear viscosity affects this evolution.

In Section II, we review the Gubser flow [22, 23] which is an exact boost-invariant solution of the relativistic Navier-Stokes equation with nontrivial radial flow velocities. In Section III, we use the so-called Zhukovsky transform to elliptically deform the Gubser flow in the transverse plane and construct an approximate solution. We then calculate in Section IV the momentum space anisotropy of this flow and discuss the connection to the empirical formula proposed by Bhalerao *et al.* [27] regarding the shear viscosity dependence of the elliptic flow. In the end, we summarise in Section V.

II. GUBSER FLOW

In this section, we briefly review the exact boost-invariant solution of the relativistic Navier-Stokes equation found by Gubser [22, 23]. The solution is naturally explained by rewriting the Minkowski metric as

$$d\hat{s}^2 = -d\hat{t}^2 + d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2 = -d\hat{\tau}^2 + d\hat{x}_\perp^2 + \hat{x}_\perp^2 d\hat{\phi}^2 + \hat{\tau}^2 d\hat{\zeta}^2, \quad (2)$$

where $\hat{\tau} \equiv \sqrt{\hat{t}^2 - \hat{z}^2}$ is the proper time and $\hat{\zeta} \equiv \tanh^{-1} \hat{z}/\hat{t}$ is the space-time rapidity. In this coordinate system, the four-velocity \hat{u}^μ of the fluid (normalized as $\hat{u}_\mu \hat{u}^\mu = -1$) reads

$$\hat{u}_\tau = -\cosh \left[\tanh^{-1} \frac{2\hat{\tau}\hat{x}_\perp}{L^2 + \hat{\tau}^2 + \hat{x}_\perp^2} \right], \quad \hat{u}_\perp = \sinh \left[\tanh^{-1} \frac{2\hat{\tau}\hat{x}_\perp}{L^2 + \hat{\tau}^2 + \hat{x}_\perp^2} \right], \quad (3)$$

with $\hat{u}_\phi = \hat{u}_\zeta = 0$ and L is roughly the initial transverse size of the fluid. The $\hat{\zeta}$ -independence of Eq. (3) and $\hat{u}_\zeta = 0$ mean that the flow expands in the longitudinal (\hat{z}) direction in a boost-invariant way. It also expands in the transverse direction with the transverse velocity

$$\hat{v}_\perp = -\frac{\hat{u}_\perp}{\hat{u}_\tau} = \frac{2\hat{\tau}\hat{x}_\perp}{L^2 + \hat{\tau}^2 + \hat{x}_\perp^2}. \quad (4)$$

Plugging this velocity profile into the relativistic Navier-Stokes equation and assuming conformal symmetry, Gubser obtained the following exact solution for the energy density

$$\hat{\mathcal{E}} = \frac{1}{\hat{\tau}^4} \frac{C^4}{(\cosh \rho)^{8/3}} \left[1 + \frac{\eta_0}{9C} (\sinh \rho)^3 {}_2F_1 \left(\frac{3}{2}, \frac{7}{6}, \frac{5}{2}; -\sinh^2 \rho \right) \right]^4, \quad (5)$$

where $C > 0$ is a constant and the shear viscosity $\hat{\eta}$ has been made dimensionless by factoring out the corresponding power of the energy density $\eta_0 \equiv \hat{\eta}/\hat{\mathcal{E}}^{3/4}$. In Eq. (5), we have defined

$$\sinh \rho \equiv -\frac{L^2 - \hat{\tau}^2 + \hat{x}_\perp^2}{2L\hat{\tau}}. \quad (6)$$

The following components of the shear tensor will be needed for a later calculation.

$$\begin{aligned} \hat{\sigma}_{\tau\perp} &= \frac{2\hat{x}_\perp(L^2 - \hat{\tau}^2 + \hat{x}_\perp^2)(L^2 + \hat{\tau}^2 + \hat{x}_\perp^2)}{3((L^2 + \hat{\tau}^2 - \hat{x}_\perp^2)^2 + (2L\hat{x}_\perp)^2)^{3/2}}, & \hat{\sigma}_{\perp\perp} &= -\frac{1}{3\hat{\tau}} \frac{(L^2 + \hat{\tau}^2 + \hat{x}_\perp^2)^2(L^2 - \hat{\tau}^2 + \hat{x}_\perp^2)}{((L^2 + \hat{\tau}^2 - \hat{x}_\perp^2)^2 + (2L\hat{x}_\perp)^2)^{3/2}}, \\ \hat{\sigma}_{\perp\phi} &= 0, & \hat{\sigma}_{\phi\phi} &= -\frac{1}{3\hat{\tau}} \frac{\hat{x}_\perp^2(L^2 - \hat{\tau}^2 + \hat{x}_\perp^2)}{\sqrt{(L^2 + \hat{\tau}^2 - \hat{x}_\perp^2)^2 + (2L\hat{x}_\perp)^2}}. \end{aligned} \quad (7)$$

As already pointed out by Gubser, the solution has a pathological behavior at large negative values of ρ corresponding to large \hat{x}_\perp and/or small $\hat{\tau}$. Indeed, when $|\sinh \rho| \gg 1$ one can approximate

$${}_2F_1 \left(\frac{3}{2}, \frac{7}{6}, \frac{5}{2}; -\sinh^2 \rho \right) \approx \frac{9}{2} (-\sinh \rho)^{-7/3} + \mathcal{O}(\sinh^{-3} \rho), \quad (8)$$

so that

$$\hat{\mathcal{E}} \approx \frac{C^4}{\hat{\tau}^4 (\cosh^2 \rho)^{4/3}} \left[1 - \frac{\eta_0}{2C} \left\{ (-\sinh \rho)^{2/3} + \mathcal{O}(1) \right\} \right]^4. \quad (9)$$

The quantity inside the square brackets is proportional to the temperature $\hat{T} \propto \hat{\mathcal{E}}^{1/4}$ and this becomes negative for sufficiently negative values of ρ . This is actually not surprising since the relativistic (first-order) Navier-Stokes equation is known to have unphysical features.¹ As demonstrated in Ref. [24], the problem can be cured by switching to the (second-order) Israel-Stewart equation. For the present purpose, we are not concerned about this issue since one can consider η_0 to be arbitrarily small (or C arbitrarily large) so that the temperature remains positive in a parametrically large region of \hat{x}_\perp and $\hat{\tau}$.

$$\frac{\hat{\tau}L}{L^2 \text{ or } \hat{x}_\perp^2} \gg \left(\frac{\eta_0}{C} \right)^{3/2}. \quad (10)$$

In this region, the solution is well-behaved and offers an attractive model for the studies of strongly interacting matter created in heavy-ion collisions as discussed in Refs. [22, 23].

¹ The problem of negative temperature also appears in an exact solution of the Navier-Stokes equation for the Bjorken flow [26].

III. ELLIPTIC SOLUTION

The Gubser solution described above is cylindrically symmetric around the \hat{z} -axis. Here we relax this restriction and construct an approximate solution which has the shape of an ellipse in the transverse plane. This can be achieved by employing the so-called Zhukovsky (Joukowski) transform² which maps a circle onto an ellipse as follows

$$\hat{x} = \left(x_{\perp} + \frac{a^2}{x_{\perp}}\right) \cos \phi = x + \frac{a^2 x}{x^2 + y^2}, \quad \hat{y} = \left(x_{\perp} - \frac{a^2}{x_{\perp}}\right) \sin \phi = y - \frac{a^2 y}{x^2 + y^2}, \quad (11)$$

where a is a constant and we only consider the region $x_{\perp} > a$. As is manifest in its complex representation $\hat{\omega} = \hat{x} + i\hat{y} = \omega + \frac{a^2}{\omega}$, Eq. (11) is a conformal transformation in two-dimensions and therefore the metric is preserved up to a Weyl factor

$$d\hat{x}^2 + d\hat{y}^2 = \left(1 - \frac{2a^2}{x_{\perp}^2} \cos 2\phi + \frac{a^4}{x_{\perp}^4}\right) (dx^2 + dy^2) \equiv A^2(dx^2 + dy^2). \quad (12)$$

Embedding this in four-dimensions, we obtain

$$\begin{aligned} d\hat{s}^2 &= -d\hat{\tau}^2 + d\hat{x}^2 + d\hat{y}^2 + \hat{\tau}^2 d\hat{\zeta}^2 \\ &= A^2 \left[-d\tau^2 + dx_{\perp}^2 + x_{\perp}^2 d\phi^2 + \tau^2 d\zeta^2 - \frac{2\tau}{A} d\tau dA - \frac{\tau^2}{A^2} (dA)^2 \right], \end{aligned} \quad (13)$$

where we have relabeled $\hat{\tau} = A\tau$ and $\hat{\zeta} = \zeta$. If the last two terms were absent, the metric inside the square brackets would be exactly Minkowskian. Let us find the conditions under which these terms can indeed be neglected. More explicitly, we find

$$dA = \frac{2a^2}{Ax_{\perp}^2} \left[\left(\cos 2\phi - \frac{a^2}{x_{\perp}^2} \right) \frac{dx_{\perp}}{x_{\perp}} + \sin 2\phi d\phi \right]. \quad (14)$$

Since the elliptic deformation $\sim \cos 2\phi$ is an $\mathcal{O}(a^2/x_{\perp}^2)$ effect, we assume $a^2/x_{\perp}^2 \ll 1$ and neglect terms of order $\mathcal{O}(a^4/x_{\perp}^4)$. Then the $(dA)^2$ term in (13) can be safely dropped. In order to drop the cross term $\mathcal{O}(d\tau dA)$ as well, we must additionally assume that $x_{\perp} \gg \tau$ and neglect terms of order $\mathcal{O}(\tau a^2/x_{\perp}^3)$ relative to the leading term.

Under these conditions, the coordinate systems $(\hat{\tau}, \hat{x}, \hat{y}, \hat{\zeta})$ and (τ, x, y, ζ) are conformally related, and one can map the Gubser solution expressed in the former coordinates into the latter.³ The transformation rule is given by

$$\mathcal{E} = A^4 \hat{\mathcal{E}}, \quad u_{\mu} = \frac{1}{A} \frac{\partial \hat{x}^{\nu}}{\partial x^{\mu}} \hat{u}_{\nu}, \quad \sigma_{\mu\nu} = \frac{1}{A} \frac{\partial \hat{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \hat{x}^{\beta}}{\partial x^{\nu}} \hat{\sigma}_{\alpha\beta}, \quad (15)$$

To the order of interest, we can approximate

$$\hat{\tau} = A\tau \approx \left(1 - \frac{a^2}{x_{\perp}^2} \cos 2\phi\right) \tau, \quad \hat{x}_{\perp} \approx x_{\perp} + \frac{a^2}{x_{\perp}} \cos 2\phi, \quad (16)$$

$$\frac{\partial \hat{\phi}}{\partial \phi} \approx 1 - \frac{2a^2}{x_{\perp}^2} \cos 2\phi, \quad \frac{\partial \hat{\phi}}{\partial x_{\perp}} \approx \frac{2a^2}{x_{\perp}^3} \sin 2\phi, \quad (17)$$

$$\sinh \rho \approx -\frac{L^2 - \tau^2 + x_{\perp}^2}{2L\tau} \left(1 + \frac{a^2(L^2 + 3x_{\perp}^2)}{x_{\perp}^2(L^2 + x_{\perp}^2)} \cos 2\phi\right). \quad (18)$$

² The Zhukovsky transform was originally used to determine the two-dimensional incompressible potential flow around an airfoil.

³ Since Eq. (11) is not an element of the Möbius transformation, it cannot exactly be promoted to a four-dimensional conformal transformation. This is why the solution obtained is an approximate one.

We thus find the energy density

$$\begin{aligned}\mathcal{E} &\approx \frac{C^4}{\tau^{4/3}} \frac{(2L)^{8/3}}{(L^4 + 2(\tau^2 + x_\perp^2)L^2 + (\tau^2 - x_\perp^2)^2)^{4/3}} \left(1 - \frac{\eta_0}{2C} \left(\frac{L^2 - \tau^2 + x_\perp^2}{2L\tau}\right)^{2/3}\right)^4 \\ &\quad \times \left(1 - \frac{a^2}{1 - \frac{\eta_0}{2C} \left(\frac{L^2 - \tau^2 + x_\perp^2}{2L\tau}\right)^{2/3}} \frac{8}{3x_\perp^2} \frac{L^2 + 3x_\perp^2}{L^2 + x_\perp^2} \cos 2\phi\right) \\ &\equiv \mathcal{E}_0 + \delta\mathcal{E}a^2 \cos 2\phi.\end{aligned}\tag{19}$$

Since $\delta\mathcal{E} < 0$, the curves of constant energy are elliptic, with the major axis in the y -direction. The flow velocity in turn is given by

$$\begin{aligned}u_\tau &\approx -1 + \mathcal{O}(\tau^2), \\ u_\perp &\approx \frac{2\tau x_\perp}{L^2 + \tau^2 + x_\perp^2} - 2\tau \left(\frac{1}{x_\perp^3} + \frac{2x_\perp}{(L^2 + x_\perp^2)^2}\right) a^2 \cos 2\phi \equiv u_{\perp 0} + \delta u_\perp a^2 \cos 2\phi, \\ u_\phi &\approx \frac{-2\tau}{x_\perp^2} \frac{L^2 + 3x_\perp^2}{L^2 + x_\perp^2} a^2 \sin 2\phi \equiv \delta u_\phi a^2 \sin 2\phi,\end{aligned}\tag{20}$$

and the shear tensor is

$$\begin{aligned}\sigma_{\perp\perp} &\approx -\frac{1}{3\tau} \left(1 - \frac{2(L^2 - 2x_\perp^2)\tau^2}{(L^2 + x_\perp^2)^2} - \frac{4\tau^2(L^4 + L^2x_\perp^2 + 6x_\perp^4)}{(L^2 + x_\perp^2)^3 x_\perp^2} a^2 \cos 2\phi\right) \equiv \sigma_{\perp\perp}^0 + \delta\sigma_{\perp\perp} a^2 \cos 2\phi, \\ \sigma_{\perp\phi} &\approx \frac{4\tau(L^2 + 3x_\perp^2)}{3x_\perp(L^2 + x_\perp^2)^2} a^2 \sin 2\phi \equiv \delta\sigma_{\perp\phi} a^2 \sin 2\phi, \\ \sigma_{\phi\phi} &\approx -\frac{x_\perp^2}{3\tau} \left(1 - \frac{2L^2\tau^2}{(L^2 + x_\perp^2)^2} + \frac{4\tau^2 L^2(L^2 + 3x_\perp^2)}{(L^2 + x_\perp^2)^3 x_\perp^2} a^2 \cos 2\phi\right) \equiv \sigma_{\phi\phi}^0 + \delta\sigma_{\phi\phi} a^2 \cos 2\phi.\end{aligned}\tag{21}$$

Note that δu_\perp is negative, meaning that both \mathcal{E} and u_\perp are stretched in the y -direction. This may seem contradictory to the standard picture that the stronger pressure (or energy density) gradient in the x -direction develops a stronger flow in the same direction. In fact, there is no contradiction. The negativeness of δu_\perp (at large x_\perp) is a direct consequence of conformal symmetry⁴ which dictates a power-law decay of \mathcal{E} and the hydrodynamic equation which, in the present accuracy, boils down to

$$\delta u_\perp = -\frac{3\tau}{8} \partial_\perp \left(\frac{\delta\mathcal{E}}{\mathcal{E}_0}\right) < 0.\tag{22}$$

(For simplicity here we consider the ideal hydrodynamic equation.) On the other hand, we find

$$\begin{aligned}\int_0^{2\pi} d\phi (u_x^2 - u_y^2) &\approx \int d\phi \left(\cos 2\phi u_\perp^2 - \frac{2\sin 2\phi}{x_\perp} u_\phi u_\perp\right) \\ &= 2\pi a^2 u_{\perp 0} \left(\delta u_\perp - \frac{\delta u_\phi}{x_\perp}\right) = \frac{16\pi a^2 \tau^2 L^2}{(L^2 + x_\perp^2)^3} > 0,\end{aligned}\tag{23}$$

where we have neglected $u_\phi^2 \sim \mathcal{O}(a^4)$. Eq. (23) shows that, due to the ϕ -component of the flow velocity which tends to point to the x -direction ($\delta u_\phi < 0$), the flow is on average stronger in the x -direction, as expected.

Eqs. (19)-(21) define an approximate elliptic solution of the Navier-Stokes equation with the shear viscosity $\eta = \eta_0 \mathcal{E}^{3/4}$. They satisfy the equation with the accuracy of order $\mathcal{O}(a^2/x_\perp^2)$ relative to the leading (isotropic) solution,

⁴ The situation is different in the confining (non-conformal) case. Suppose that the energy density decays as a Gaussian instead of a power-law

$$\mathcal{E} \sim e^{-\frac{x_\perp^2}{\sigma_x^2} - \frac{y^2}{\sigma_y^2}} \approx \exp\left(-\frac{x_\perp^2}{2} \left(\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2}\right)\right) \left(1 - \frac{x_\perp^2}{2} \left(\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2}\right) \cos 2\phi\right),$$

where $\sigma_y > \sigma_x$. The Euler equation is then

$$u_\perp \sim -\tau \partial_\perp \ln \mathcal{E} \sim \tau x_\perp \left(\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2}\right) \cos 2\phi = \delta u_\perp \cos 2\phi,$$

for the ϕ -dependent part. Thus δu_\perp is positive in this case.

but break down at order $\mathcal{O}(a^2\tau^2/x_\perp^4)$ which can be checked explicitly.⁵ One way to understand this is to notice that there is in fact an $\mathcal{O}(\tau^2/x_\perp^2)$ uncertainty in the definition of a^2 . Indeed, we could have employed a nonconstant $a^2(\tau)$ already in Eq. (11) as long as its τ -dependence is sufficiently weak

$$\frac{1}{a^2} \frac{\partial a^2}{\partial \tau} \lesssim \mathcal{O}\left(\frac{\tau}{x_\perp^2}\right). \quad (24)$$

If Eq. (24) is satisfied, the extra terms that would appear in the transformation as in Eq. (13) are of the same order as, or less than those already neglected. The maximal uncertainty incurred by Eq. (24) is

$$a^2(\tau) \sim a_0^2 \left(1 + \mathcal{O}\left(\frac{\tau^2}{x_\perp^2}\right)\right), \quad (25)$$

and this affects the Navier-Stokes equation at order $\mathcal{O}(a^2\tau^2/x_\perp^4)$.

A similar caveat applies to the eccentricity of the flow which we define as

$$\epsilon(\tau) \equiv \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle} \equiv \frac{\int dx dy (y^2 - x^2) \mathcal{E}(x_\perp, \tau)}{\int dx dy (y^2 + x^2) \mathcal{E}(x_\perp, \tau)}. \quad (26)$$

For the ideal solution at early times, we find

$$\epsilon^{ideal}(\tau \approx 0) \approx \frac{14a^2}{9L^2}. \quad (27)$$

On general grounds, one expects that $\epsilon(\tau)$ decreases with time. Unfortunately, this τ -dependence is not reliably calculable in the present approach due to the above uncertainty in a^2 . We however note that Eq. (25) suggests that $\epsilon(\tau)$ decreases quadratically in τ

$$\epsilon(\tau) \sim \epsilon_0 \left(1 - c \frac{\tau^2}{L^2}\right), \quad (28)$$

where $c > 0$ remains undetermined. Eq. (28) appears to be consistent with the result of numerical simulations [9, 14] (see also, [28]).

Before proceeding to the calculation of elliptic flow with viscous corrections, we make one more simplification. In order to clearly see the role of the viscosity, we shall neglect the terms of order $\mathcal{O}(\tau^2/x_\perp^2)$ and $\mathcal{O}(\tau^2/L^2)$ in the energy density Eq. (19) while keeping powers of $\eta_0(L/\tau)^{2/3}$ up to quadratic order. Actually, once we go beyond linear order in η_0 , we must be careful with the $\mathcal{O}(1)$ terms inside the brackets of Eq. (9). The condition that these terms are parametrically smaller than the $\mathcal{O}(\eta_0^2(L/\tau)^{4/3})$ term is

$$\left(\frac{\eta_0}{C}\right)^{3/4} \gg \frac{\tau}{L \text{ or } x_\perp} \gg \left(\frac{\eta_0}{C}\right)^{3/2}, \quad (29)$$

where another constraint in Eq. (10) has been included. Thus, strictly speaking, the following analysis is valid in the window as shown in Eq. (29) in the viscous case with $\eta_0 \neq 0$.

IV. VISCOUS EFFECT ON ELLIPTIC FLOW

We now turn to the discussion of the elliptic flow parameter v_2 which characterizes the momentum space anisotropy in the final state. Besides the leading contribution from the ideal flow, we also include the shear viscous correction which plays an important role in the phenomenological study of quark gluon plasma [11, 13–15, 17].

With an analytic solution of the Navier-Stokes equation at hand, we can get information about v_2 via a closely related quantity [8, 9]

$$\epsilon_p(\tau) \equiv \frac{\int dx dy (T_{xx} - T_{yy})}{\int dx dy (T_{xx} + T_{yy})}. \quad (30)$$

⁵ Note that we have already neglected terms of order $\mathcal{O}(\tau^2/x_\perp^2)$ in the $\mathcal{O}(a^2)$ corrections in Eqs. (19)-(21).

Explicitly, we find

$$T_{xx} + T_{yy} \approx \frac{2\mathcal{E}}{3} (1 + 2u_{\perp}^2) + \pi_{\perp\perp} + \frac{1}{x_{\perp}^2} \pi_{\phi\phi}, \quad (31)$$

$$\begin{aligned} T_{xx} - T_{yy} &= \frac{4\mathcal{E}}{3} (u_x^2 - u_y^2) + \pi_{xx} - \pi_{yy} \\ &\approx \cos 2\phi \left(\frac{4\mathcal{E}}{3} u_{\perp}^2 + \pi_{\perp\perp} - \frac{1}{x_{\perp}^2} \pi_{\phi\phi} \right) - \frac{2 \sin 2\phi}{x_{\perp}} \left(\frac{4\mathcal{E}}{3} u_{\perp} u_{\phi} + \pi_{\perp\phi} \right), \end{aligned} \quad (32)$$

where $\pi^{\mu\nu}$ is the shear-stress tensor. Unlike v_2 , $\epsilon_p(\tau)$ in Eq. (30) is defined at all times, and its value at the ‘build-up time’

$$\tau_f \sim \frac{L}{c_s}, \quad (33)$$

where L represents the characteristic transverse size and c_s is the speed of sound, is considered to be a measure of $v_2 \propto \epsilon_p(\tau_f)$. ($c_s = 1/\sqrt{3}$ in a conformal theory.) However, as already mentioned, $\tau_f \sim \mathcal{O}(L)$ is outside the region of validity for our approximate solution. Nevertheless, we show that Eq. (30) evaluated at $\tau \ll \tau_f$ provides us with useful analytical insights into the property of $\epsilon_p(\tau)$ as we extrapolate $\tau \rightarrow \tau_f$.

First, let us consider the denominator in Eq. (31). Since $\epsilon_p \propto a^2$ and the numerator is $\mathcal{O}(a^2)$, we can set $a = 0$ in the denominator from the beginning. We then recall that in the Navier-Stokes approximation,

$$\pi_{\mu\nu} = -2\eta\sigma_{\mu\nu} = -2\eta_0\mathcal{E}^{3/4}\sigma_{\mu\nu}, \quad (34)$$

where $\sigma_{\mu\nu}$ is given by Eq. (21) and

$$\begin{aligned} \mathcal{E}^{3/4} &\approx \frac{C^3}{\tau} \frac{(2L)^2}{(L^2 + x_{\perp}^2)^2} \left(1 - \frac{\eta_0}{2C} \left(\frac{L^2 + x_{\perp}^2}{2L\tau} \right)^{2/3} \right)^3 \\ &\quad \times \left(1 - \frac{a^2}{1 - \frac{\eta_0}{2C} \left(\frac{L^2 + x_{\perp}^2}{2L\tau} \right)^{2/3}} \frac{2}{x_{\perp}^2} \frac{L^2 + 3x_{\perp}^2}{L^2 + x_{\perp}^2} \cos 2\phi \right) \\ &\equiv \mathcal{E}_0^{3/4} + \delta\mathcal{E}^{3/4} a^2 \cos 2\phi, \end{aligned} \quad (35)$$

in the present approximation. Moreover, $u_{\perp}^2 \sim \mathcal{O}(\tau^2)$ in the factor $1 + 2u_{\perp}^2$ can be neglected. It is then straightforward to show that

$$\int dxdy (T_{xx} + T_{yy}) \approx \frac{8\pi C^4}{5\tau^{4/3}} \left(\frac{2}{L} \right)^{2/3} \left(1 - \frac{15\eta_0^2}{2C^2} \left(\frac{L}{2\tau} \right)^{4/3} \right). \quad (36)$$

Note that the viscous correction shows up only at $\mathcal{O}(\eta_0^2)$.⁶

We now turn to the numerator which requires some extra care. After the ϕ integral, we get

$$\begin{aligned} \int d\phi (T_{xx} - T_{yy}) &= \frac{4\pi a^2 u_{\perp 0}}{3} \left\{ \delta\mathcal{E} u_{\perp 0} + 2\mathcal{E}_0 \left(\delta u_{\perp} - \frac{1}{x_{\perp}} \delta u_{\phi} \right) \right\} \\ &\quad - 2\pi a^2 \eta_0 \left\{ \left(\sigma_{\perp\perp}^0 - \frac{\sigma_{\phi\phi}^0}{x_{\perp}^2} \right) \delta\mathcal{E}^{3/4} + \left(\delta\sigma_{\perp\perp} - \frac{\delta\sigma_{\phi\phi}}{x_{\perp}^2} - \frac{2\delta\sigma_{\perp\phi}}{x_{\perp}} \right) \mathcal{E}_0^{3/4} \right\} \\ &= \frac{16\pi a^2 \tau^2}{3(L^2 + x_{\perp}^2)^2} \left(x_{\perp}^2 \delta\mathcal{E} + \frac{4L^2 \mathcal{E}_0}{L^2 + x_{\perp}^2} \right) + \frac{8\pi a^2 \eta_0 \tau}{3(L^2 + x_{\perp}^2)^2} \left(x_{\perp}^2 \delta\mathcal{E}^{3/4} + \frac{4L^2 \mathcal{E}_0^{3/4}}{L^2 + x_{\perp}^2} \right), \end{aligned} \quad (37)$$

where the potentially dangerous $\mathcal{O}(1/x_{\perp}^2)$ terms in Eqs. (20) and (21) which could cause trouble in the remaining dx_{\perp}^2 integral have canceled out. In Eq. (37), we recognize two types of contributions with clear but distinct physical

⁶ In fact, the $\mathcal{O}(\eta_0^3)$ term of the denominator (but not the numerator) is divergent because the x_{\perp} -integral does not converge as $x_{\perp} \rightarrow \infty$. This is an artifact of the solution Eq. (9) which becomes unphysical at large x_{\perp} , and limits our calculation of ϵ_p to $\mathcal{O}(\eta_0^2)$.

interpretations. The terms proportional to $\delta\mathcal{E}$ and $\delta\mathcal{E}^{3/4}$ are due to the spatial anisotropy of the source (energy density). Since the bulk of matter is initially stretched in the y -direction ($\delta\mathcal{E} < 0$), these terms give a negative contribution to ϵ_p . On the other hand, the terms proportional to \mathcal{E}_0 and $\mathcal{E}_0^{3/4}$ are due to the anisotropy of the flow velocity δu . As already discussed in Eq. (23), they give a positive contribution to ϵ_p .

We thus evaluate the two contributions separately and find, after dividing by Eq. (36),

$$\left. \frac{\int dx dy (T_{xx} - T_{yy})}{\int dx dy (T_{xx} + T_{yy})} \right|_{\delta\mathcal{E}} = \frac{20a^2\tau^2}{3L^4} \left[-\frac{80}{77} + \frac{3\eta_0}{2C} \left(\frac{L}{2\tau} \right)^{2/3} - \frac{3264\eta_0^2}{385C^2} \left(\frac{L}{2\tau} \right)^{4/3} \right], \quad (38)$$

and

$$\left. \frac{\int dx dy (T_{xx} - T_{yy})}{\int dx dy (T_{xx} + T_{yy})} \right|_{\delta u} = \frac{20a^2\tau^2}{3L^4} \left[\frac{6}{7} - \frac{3\eta_0}{2C} \left(\frac{L}{2\tau} \right)^{2/3} + \frac{513\eta_0^2}{70C^2} \left(\frac{L}{2\tau} \right)^{4/3} \right]. \quad (39)$$

Summing the two contributions, we arrive at

$$\epsilon_p(\tau) = \frac{20a^2\tau^2}{3L^4} \left[-\frac{2}{11} - \frac{177\eta_0^2}{154C^2} \left(\frac{L}{2\tau} \right)^{4/3} \right]. \quad (40)$$

Surprisingly, ϵ_p is negative as a result of the slightly larger contribution from the source anisotropy. This is at odds with the observed behavior in hydrodynamic simulations [8, 12, 14] which is qualitatively quite consistent with the contribution from the flow anisotropy (39) *alone*, namely, $\epsilon_p(\tau)$ is positive, grows quadratically in time $\epsilon_p(\tau) \sim \tau^2$, and the viscosity tends to suppress it.⁷

While this discrepancy may seem worrisome, one should notice the large cancelation which resulted in a barely negative value found in (40). This suggests that whether the source anisotropy contribution is large enough to flip the sign of ϵ_p is subtle and model-dependent. It is then interesting that our conformal solution reveals and exemplifies the logical possibility that even if $\int d^2x_\perp (u_x^2 - u_y^2)$ is positive, $\int d^2x_\perp \mathcal{E}(u_x^2 - u_y^2)$ can become negative. When this occurs, the simple proportionality between v_2 and ϵ_p is far from obvious and may be subject to large ‘non-flow’ effects (see e.g., Refs. [16, 29]).

What are, then, the main characteristics of a given model which determine the sign of ϵ_p ? In the model at hand, the transverse flow $u_x^2 - u_y^2$ generated by the pressure gradient is weak and insufficient to make ϵ_p positive because δu_\perp is negative (cf., (23)). This is an artifact of conformal symmetry which dictates that the energy density should decay as a power-law. In confining theories including QCD, δu_\perp will be positive (see Footnote 4), and one therefore expects that the flow anisotropy contribution (39) dominates over the source anisotropy contribution (38). After all, elliptic flow is the anisotropy in the flow velocity, and this is faithfully incorporated in (39), but not in (38).⁸

Let us therefore take a closer look at Eq. (39). Parametrically, and to linear order in η_0 , this can be rewritten as

$$\left. \frac{\epsilon_p(\tau)}{\epsilon_p^{ideal}(\tau)} \right|_{\delta u} \sim \frac{1}{1 + \frac{\eta_0}{C} \left(\frac{L}{\tau} \right)^{2/3}} \sim \frac{1}{1 + \frac{\eta L^2}{C^3 \mathcal{E}^{1/4}}} \sim \frac{1}{1 + \frac{L^2}{\sigma dN/dY}}, \quad (41)$$

where we have used $\mathcal{E} \sim T^4 \sim C^4/(\tau^{4/3}L^{8/3})$ and the kinetic theory relation $\eta/\mathcal{E}^{1/4} \sim \eta/T \sim 1/\sigma$ (with σ being the cross section of microscopic degrees of freedom) together with an estimate $C^3 \sim dN/dY$ of the rapidity distribution of particle multiplicity in the final state (see, Eq. (44) of Ref. [22]). Now let us compare Eq. (41) with the following empirical formula proposed in heavy-ion collisions [27] (and also in high-multiplicity proton-proton collisions [33])

$$\frac{v_2}{v_2^{ideal}} = \frac{1}{1 + \frac{K}{K_0}}, \quad K \equiv \frac{S_\perp}{c_s \sigma \frac{dN}{dY}}, \quad (42)$$

where $K_0 > 0$ is a number of order unity and S_\perp is the initial transverse overlap area of the two colliding nuclei. Despite its simplicity, the formula in Eq. (42) is quite successful in fitting the RHIC/LHC data and the results of

⁷ Readers may wonder whether the results in Eqs. (38)–(40), which are of the order of $\mathcal{O}(a^2\tau^2/L^4)$, are reliable in view of the difficulty we have encountered at this order when analyzing the hydrodynamic equation and the eccentricity. However, in the previous examples, the uncertainty at order $\mathcal{O}(a^2\tau^2/L^4)$ stems from unknown corrections to the lower order $\mathcal{O}(a^2/L^2)$ results. In contrast, Eqs. (38)–(40) are the leading order results for the momentum anisotropy. The would-be lower order terms of order $\mathcal{O}(\tau^2/L^2)$ and $\mathcal{O}(a^2/L^2)$, as well as their uncertainty have been canceled. It is also straightforward to see that the genuine $\mathcal{O}(a^2\tau^2/L^4)$ corrections of \mathcal{E} and u^μ can only enter at even higher order in the results in Eqs. (38)–(40).

⁸ Another difference from hydrodynamic simulations is the treatment of the initial velocity. In typical numerical simulations, the transverse velocity at the initial time $\tau = \tau_0$ is set to be zero, whereas in our solution the velocity (20) is nonzero for any $\tau_0 > 0$. Simulations with nonvanishing $u_\perp(\tau_0)$ do exist (see, e.g., [30–32]), but so far they only studied v_2 in the final state. It would be interesting to see the effect of the initial velocity on the early-stage development of ϵ_p in these simulations.

viscous hydrodynamic simulations [12, 27, 34]. With the natural identification $S_{\perp} = L^2$, parametrically the agreement between Eqs. (41) and (42) is perfect.⁹ Thus we have presented an explicit justification of the empirical formula Eq. (42) to linear order in the ‘Knudsen’ number $K \sim \eta_0(L/\tau)^{2/3}$. At higher order in K , our result suggests a power series with alternating sign (remember that $\mathcal{E} \sim (1 - K)^4$ before the d^2x_{\perp} integration). Since this is not a geometric series, we expect a deviation from the formula Eq. (42) starting from $\mathcal{O}(K^2)$. Nevertheless, Eq. (42) does capture the main feature of the series and this is probably the reason of its successful applications in phenomenology.

V. CONCLUSION

In conclusion, we have constructed an approximate boost-invariant elliptic solution of the Navier-Stokes equation with a well-defined region of validity. In this region, the leading contribution to the momentum anisotropy (30) of the fluid is analytically calculated. Driven by the spatial anisotropy, a stronger flow develops in the direction of larger pressure gradient. While this scenario is well-known and well-tested numerically, in the presence of viscosity it has not been demonstrated with the level of analytical precision we have been able to present in this paper. We also pointed out the potentially large negative contribution to ϵ_p due to the source anisotropy which, depending on models, can even flip the sign of ϵ_p . Finally, by focusing on the flow anisotropy contribution in Eq. (39), we have discussed the connection with the empirical formula (42) previously suggested in heavy-ion phenomenology.

Acknowledgements

We are grateful to Jorge Noronha and Jean-Yves Ollitrault for many helpful discussions. We also thank Tetsufumi Hirano and Guang-You Qin for discussions and comments.

-
- [1] K. H. Ackermann *et al.* [STAR Collaboration], Phys. Rev. Lett. **86**, 402 (2001) [nucl-ex/0009011].
 - [2] K. Adcox *et al.* [PHENIX Collaboration], Phys. Rev. Lett. **89**, 212301 (2002) [nucl-ex/0204005].
 - [3] B. B. Back *et al.* [PHOBOS Collaboration], Phys. Rev. Lett. **89**, 222301 (2002) [nucl-ex/0205021].
 - [4] K. Aamodt *et al.* [ALICE Collaboration], Phys. Rev. Lett. **105**, 252302 (2010) [arXiv:1011.3914 [nucl-ex]].
 - [5] G. Aad *et al.* [ATLAS Collaboration], Phys. Lett. B **707**, 330 (2012) [arXiv:1108.6018 [hep-ex]].
 - [6] S. Chatrchyan *et al.* [CMS Collaboration], Eur. Phys. J. C **72**, 2012 (2012) [arXiv:1201.3158 [nucl-ex]].
 - [7] J. -Y. Ollitrault, Phys. Rev. D **46**, 229 (1992).
 - [8] P. F. Kolb, J. Sollfrank and U. W. Heinz, Phys. Lett. B **459**, 667 (1999) [nucl-th/9906003].
 - [9] P. F. Kolb, J. Sollfrank, P. V. Ruuskanen and U. W. Heinz, Nucl. Phys. A **661**, 349 (1999) [nucl-th/9907025].
 - [10] P. Huovinen, P. F. Kolb, U. W. Heinz, P. V. Ruuskanen and S. A. Voloshin, Phys. Lett. B **503**, 58 (2001) [hep-ph/0101136].
 - [11] D. Teaney, Phys. Rev. C **68**, 034913 (2003) [nucl-th/0301099].
 - [12] C. Gombeaud and J. -Y. Ollitrault, Phys. Rev. C **77**, 054904 (2008) [nucl-th/0702075].
 - [13] Z. Xu, C. Greiner and H. Stöcker, Phys. Rev. Lett. **101**, 082302 (2008) [arXiv:0711.0961 [nucl-th]].
 - [14] M. Luzum and P. Romatschke, Phys. Rev. C **78**, 034915 (2008) [Erratum-ibid. C **79**, 039903 (2009)] [arXiv:0804.4015 [nucl-th]].
 - [15] H. Song and U. W. Heinz, J. Phys. G **36**, 064033 (2009) [arXiv:0812.4274 [nucl-th]].
 - [16] T. Hirano, P. Huovinen and Y. Nara, Phys. Rev. C **83**, 021902 (2011) [arXiv:1010.6222 [nucl-th]].
 - [17] H. Niemi, G. S. Denicol, P. Huovinen, E. Molnar and D. H. Rischke, Phys. Rev. Lett. **106**, 212302 (2011) [arXiv:1101.2442 [nucl-th]].
 - [18] T. Csorgo, L. P. Csernai, Y. Hama and T. Kodama, Heavy Ion Phys. A **21**, 73 (2004) [nucl-th/0306004].
 - [19] M. Csanad, T. Csorgo and B. Lorstad, Nucl. Phys. A **742**, 80 (2004) [nucl-th/0310040].
 - [20] S. -J. Sin, S. Nakamura and S. P. Kim, JHEP **0612**, 075 (2006) [hep-th/0610113].
 - [21] R. Peschanski and E. N. Saridakis, Phys. Rev. C **80**, 024907 (2009) [arXiv:0906.0941 [nucl-th]].
 - [22] S. S. Gubser, Phys. Rev. D **82**, 085027 (2010) [arXiv:1006.0006 [hep-th]].
 - [23] S. S. Gubser and A. Yarom, Nucl. Phys. B **846**, 469 (2011) [arXiv:1012.1314 [hep-th]].
 - [24] H. Marrochio, J. Noronha, G. S. Denicol, M. Luzum, S. Jeon and C. Gale, arXiv:1307.6130 [nucl-th].
 - [25] Y. Hatta, J. Noronha and B. -W. Xiao, Phys. Rev. D **89**, 051702 (2014) [arXiv:1401.6248 [hep-th]].

⁹ The factor of c_s in (42) originates from the build-up time Eq. (33) of the elliptic flow. Our solution, valid for $\tau \ll \tau_f$, does not have the accuracy to say anything about this $\mathcal{O}(1)$ factor.

- [26] Y. Hatta, J. Noronha and B. -W. Xiao, arXiv:1403.7693 [hep-th].
- [27] R. S. Bhalerao, J. -P. Blaizot, N. Borghini and J. -Y. Ollitrault, Phys. Lett. B **627**, 49 (2005) [nucl-th/0508009].
- [28] A. Taliotis, JHEP **1009**, 102 (2010) [arXiv:1004.3500 [hep-th]].
- [29] P. Huovinen, P. F. Kolb and U. W. Heinz, Nucl. Phys. A **698**, 475 (2002) [nucl-th/0104020].
- [30] G. -Y. Qin, H. Petersen, S. A. Bass and B. Muller, Phys. Rev. C **82**, 064903 (2010) [arXiv:1009.1847 [nucl-th]].
- [31] L. Pang, Q. Wang and X. -N. Wang, Phys. Rev. C **86**, 024911 (2012) [arXiv:1205.5019 [nucl-th]].
- [32] C. Gale, S. Jeon, B. Schenke, P. Tribedy and R. Venugopalan, Phys. Rev. Lett. **110**, 012302 (2013) [arXiv:1209.6330 [nucl-th]].
- [33] E. Avsar, C. Flensburg, Y. Hatta, J. -Y. Ollitrault and T. Ueda, Phys. Lett. B **702**, 394 (2011) [arXiv:1009.5643 [hep-ph]].
- [34] H. -J. Drescher, A. Dumitru, C. Gombeaud and J. -Y. Ollitrault, Phys. Rev. C **76**, 024905 (2007) [arXiv:0704.3553 [nucl-th]].